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LETTER TO THE EDITOR

Scaling at the percolation threshold above six dimensions

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Abstract. The fractal dimensionality of the infinite cluster at the percolation threshold for dimensionalities $d>6$ is shown to be $D=4$ (rather than the naive finite size scaling prediction $D=d-2$). Similarly, the conductivity of a sample of size L scales as L^{-d} (rather than L^{-6}). This anomalous behaviour is related to a dangerous irrelevant variable, associated with the probability to have vertices of three bonds. The crossover to the 'homogeneous' behaviour occurs at length scales which are short compared with the correlation length. The 'links and blobs' picture is confirmed for $d>6$, and the size of the latter is estimated.

Much of the current interest in the properties of dilute systems concentrates on their geometrical structure in the vicinity of the percolation threshold, p_c. (For detailed reviews, see e.g. Stauffer 1979, Essam 1980, Kirkpatrick 1979). As the concentration p approaches p_c, the pair connectedness length ξ diverges, $\xi \propto |p-p_c|^{-\nu}$. It is generally believed that on large length scales, $L \gg \xi$, the *infinite cluster* which appears for $p>p_c$ is *homogeneous*, with site (or bond) density $P_\infty \propto (p-p_c)^\beta \propto \xi^{-\beta/\nu}$. This homogeneity is believed to disappear for shorter length scales, $L<\xi$. For these scales, the infinite cluster is believed to be *self-similar*, with a typical fractal *dimensionality D*. The value of D has been the subject of many recent papers (e.g. Stanley 1977, 1981, Pike and Stanley 1981, Stauffer 1980, Kirkpatrick 1979, Gefen *et al* 1981, Mandelbrot 1977, 1982, Kapitulnik *et al* 1983). To define D, consider a point on the infinite cluster and count the number $M(L)$ of points on the same cluster within a volume L^d (of linear size L in d dimensions) centred at that point. Self-similarity implies that (Mandelbrot 1977, 1982, Kapitulnik *et al* 1983 and references)

$$M(L) \propto L^D, \qquad \underline{a} \ll L \ll \xi, \tag{1}$$

where $\underline{a}$ is a microscopic typical length (e.g. the lattice constant).

For $L \gg \xi$, homogeneity implies that $M(L) \propto P_\infty L^d \propto \xi^{-\beta/\nu} L^d$. Assuming that ξ is the only relevant length in the problem, we may write $M(L,\xi)$ in the *scaling form* (e.g. Kapitulnik *et al* 1983)

$$M(L,\xi) = \xi^{-\beta/\nu} L^d m(L/\xi). \tag{2}$$

For $L \ll \xi$, M should become independent of ξ. Thus, $m(x) \propto x^{-\beta/\nu}$ and $M(L) \propto L^{d-\beta/\nu}$, i.e.

$$D = d-\beta/\nu. \tag{3}$$

This result also follows from finite size scaling at p_c (Kirkpatrick 1979), and has been confirmed by direct Monte Carlo and laboratory experiments at $d=2$ (e.g. Kapitulnik *et al* 1983, Kapitulnik and Deutscher 1982).

Hyperscaling relations among the critical exponents, e.g. $d\nu = 2\beta + \gamma$, imply that (3) may be replaced by

$$D = (\beta + \gamma)/\nu, \tag{4}$$

where γ describes the divergence of the mean squared size of the clusters.

An alternative way to derive (4) is as follows: the number of sites, $s(\xi)$, in a finite cluster of size ξ, scales as $|p - p_c|^{-1/\sigma}$, where $1/\sigma = \beta + \gamma$ is the 'magnetic' exponent (Stauffer 1979). If one assumes that $s(\xi) \propto \xi^D$ ('strong self-similarity'), this implies equation (4) (Stanley and Coniglio 1983). Note that the exponent σ is truly 'thermodynamic', and does not involve hyperscaling.

Above six dimensions the critical exponents are known to assume their mean field values, $\beta = 1$, $\gamma = 1$, $\nu = \frac{1}{2}$ (Toulouse 1974, Harris *et al* 1975). Thus, (3) predicts that $D = d - 2$, while (4) yields $D = 4$. The aim of the present paper is to understand this discrepancy (which is absent for $d < 6$). We find that $D = 4$ for all $d > 6$ (with logarithmic corrections at $d = 6$), and relate the breakdown of (3) to the appearance of an *additional important length* in the problem, L_w. If w denotes the probability to find vertices at which three bonds meet, then one possible definition of this new length arises via $w^2 L_w^{6-d} = 1$. Equation (2) must now be replaced by

$$M(L, \xi, L_w) = w^{-1} \xi^{-2} L^d \tilde{m}(L/\xi, L_w/\xi). \tag{5}$$

For $L_w \ll L \ll \xi$ this reduces to $M \propto w^{-1} \xi^{-2} L^d (L/\xi)^{4-d} (\xi/L_w)^{6-d} \propto wL^4$. For $L \gg \xi$ we reproduce the homogeneous result, $M \propto P_\infty L^d \propto w^{-1} \xi^{-2} L^d$. The appearance of the additional length, L_w, and the breakdown of the hyperscaling result (3) for $d > 6$, are related to the fact that w plays the role of a '*dangerous irrelevant variable*' (Fisher 1973). A similar breakdown of finite size scaling was recently noted by Brézin (1982) for the Ising model at $d > 4$.

Another exponent of interest concerns the behaviour of the *conductivity of dilute resistor networks*, Σ. Near p_c, in the homogeneous regime $L \gg \xi$, one has $\Sigma \propto (p - p_c)^\mu \propto \xi^{-\mu/\nu}$. Therefore, the conductance of a sample of linear size L behaves as

$$g(L) \propto L^{d-2} \Sigma(L) \propto L^{d-2} \xi^{-\mu/\nu}. \tag{6}$$

Scaling with ξ as the only length would imply that

$$\Sigma(L, \xi) = \xi^{-\mu/\nu} S(L/\xi), \tag{7}$$

approaching $\Sigma \propto L^{-\mu/\nu}$ (and $g \propto L^{d-2-\mu/\nu}$) in the self-similar regime $L \ll \xi$ (Gefen *et al* 1981 and references). For $d > 6$ one has $\mu = 3$ (de Gennes 1976), so that this scaling implies that $g(L) \propto L^{d-8}$. Instead, we argue below that

$$g(L) \propto L^{-2} \tag{8}$$

for all $d > 6$, $L \ll \xi$, and therefore that $\Sigma(L)$ crosses over from L^{-d} at small L to $w^{-2} \xi^{-6}$ at large L.

The result (8) has important consequences with regard to the *localisation of electrons* at p_c: since $d(\ln g)/d(\ln L)$ is negative, *all the electronic states must be localised at* p_c (Gefen *et al* 1983) for *all d*! The detailed crossover from $g \propto L^{-2}$ to $g \propto L^{d-2} \xi^{-6}$ should identify the microscopic conductance (for $p > p_c$) at which extended states will begin to appear.

We now give some more details on our arguments. We begin our discussion with a heuristic argument, which sheds light on the geometrical structure of the infinite cluster at $d > 6$. This argument is based on the 'links and nodes' model (Skal and

Shklovskii 1974, de Gennes 1976), which is expected to be valid for $d>6$ (Gefen 1982). In this model, the backbone of the infinite cluster is composed of quasi-one-dimensional links, which connect nodes. At high dimensions, the links are expected to behave as random walks, so that the actual number of sites on a backbone link at scale L is of order $M_{\mathrm{B}}(L)\propto L^2$. A fraction of these sites, proportional to w, have 'dangling bonds' attached to them. As we shall show below, the average 'mass' $M_{\mathrm{d}}(L)$ of such a dangling bond is also of order L^2. Thus

$$M(L)=wM_{\mathrm{B}}(L)M_{\mathrm{d}}(L)\propto wL^4. \tag{9}$$

We now turn to the homogeneous regime ($L\gg\xi$). Here, we expect that $M(L)=L^dP_\infty$, and that $M_{\mathrm{B}}(L)=L^dP_{\mathrm{B}}$, where P_{B} is the probability per site to belong to the backbone. A site belongs to the backbone if there exist at least two independent routes from it to infinity. If the probability of such a route is R, then to leading order in R $P_\infty\propto \mathrm{R}$, $\mathrm{P_B}\propto R^2$ and thus $P_{\mathrm{B}}\propto P_\infty^2$. This result is exact on Cayley trees (Stephen, private communication, Harris and Lubensky 1983, Stein 1983), and is correct for all $d>6$ (when loops are irrelevant). For $L\gg\xi$, we expect the size of the dangling bonds to be cut off at $L\sim\xi$, so that $M_{\mathrm{d}}(L)$ is replaced by ξ^2 and $L^dP_\infty=M(L)=wM_{\mathrm{B}}(L)M_{\mathrm{d}}(L)\propto wL^dP_\infty^2\xi^2$. Therefore, $P_\infty\propto 1/(w\xi^2)$. The same result follows from mean field theory (see below).

To show that $M_{\mathrm{d}}(L)\propto L^2$, we start with the probability that a site belongs to a finite cluster of size s, sn_s. At $p=p_{\mathrm{c}}$, one has the power law behaviour $sn_s\propto s^{1-\tau}$ (Stauffer 1979). For $d>6$, $\tau=2+1/\delta=5/2$, *independent of d.* Moreover, mean field theory yields $sn_s\propto w^{-1/2}\,\mathrm{s}^{-3/2}$ (Stephen 1977). This is equal to the Laplace transform of the equation of state, $P_\infty(h)$, where h is the 'ghost' field. Since this equation involves only 'thermodynamic' exponents (e.g. β, γ, δ), the result $\tau=5/2$ does not depend on hyperscaling.

Equation (9) used the fact that the backbone cuts each 'dangling bond' only once. Assuming that the distribution of the dangling bonds is the same as that of the finite clusters (Gefen 1982), $M_{\mathrm{d}}(L)$ is equal to the average size of all the finite clusters with less than $M(L)$ sites,

$$M_{\mathrm{d}}(L)\propto\int_1^{M(L)} sn_s s\,\mathrm{d}s\propto\int_1^{M(L)} w^{-1/2}\,\mathrm{s}^{-1/2}\,\mathrm{d}s\propto \mathrm{w}^{-1/2}\,M(L)^{1/2}.$$

Using the LHS of (9), this becomes $M(L)\propto \mathrm{w}^{1/2}M_{\mathrm{B}}(\mathrm{L})M(L)^{1/2}$, and we find

$$M(L)\propto wM_{\mathrm{B}}(L)^2. \tag{10}$$

Substituting $M_{\mathrm{B}}(L)\propto L^2$ we reproduce the RHS of (9) and we confirm that $M_{\mathrm{d}}(L)\propto L^2$.

In the above argument we assumed that the backbone cuts each 'dangling bond' only once. In a volume of linear size L, the number of sites on the infinite cluster is $M(L)\propto wL^4$, and the number of sites on the backbone is $M_{\mathrm{B}}(L)\propto L^2$. The number of possible 'meetings' between them is thus of order $wM(L)M_{\mathrm{B}}(L)$,† i.e. w^2L^6, and the density of such vertices is w^2L^{6-d}. (Alternatively, this is the density of 'meetings' between the 'full' infinite cluster, of mass wL^4, and an 'average' dangling bond, of 'mass' L^2.) For $d>6$ we see that this renormalised density decreases with increasing L, and therefore our argument is consistent at large L. The theory must be modified for $d<6$.

Note also that the density of vertices becomes smaller than unity for $L>L_w$. This explains the physical meaning of L_w: the vertices are in fact dense 'blobs' of size L_w,

† Each meeting creates a new vertex, hence the factor w.

and the geometrical picture used above applies only for $L > L_w$. The 'links and nodes' picture is thus turned into the 'links, nodes and blobs' picture, due to Stanley and Coniglio (1983): the quasi-one-dimensional links meet at multi-bond dense 'blobs', of typical size L_w, and the Stanley–Coniglio picture is quantitatively confirmed for $d < 6$. Note that for $d < 6$ there exists no length except ξ, and the 'blobs' (if they can be defined at all) are of typical size ξ. The model should then probably be replaced by a self-similar fractal (Gefen *et al* 1981).

Having shown the consistency of our geometrical picture for $d > 6$, equation (8) now follows immediately: the resistance between two points at linear distance L is equal to that of the backbone link between them, which has $M_B(L) \propto L^2$ basic resistors in series. Thus $g(L) \propto 1/M_B(L) \propto L^{-2}$, for all $d > 6$.

A more explicit calculation of $M(L)$ considers the conditional probability $\rho_s(r)$ that a site at a distance r from the origin belongs to a cluster of s sites, given that the origin belongs to it. The percolation *correlation function* $G(r)$ is given by (Essam 1980)

$$G(r) = \sum_{s=1}^{\infty} sn_s\rho_s(r) + P_\infty\rho_\infty(r) - P_\infty^2. \tag{11}$$

The function $G(\mathrm{r})$ is explicitly known. It has the general scaling form $G(r) = r^{2-d-\eta}G(r/\xi)$. For $d > 6$, $\eta = 0$ and $G(r)$ is simply proportional to the Fourier transform of $(k^2 + r_0)^{-1}$, with $r_0 \propto (p - p_c)$ (Aharony 1980). At $p = p_c$, one thus has $G(r) \propto 1/r^{d-2}$.

Assuming that $M(L) \propto w^xL^D$, 'strong self-similarity' (Wilke *et al* 1984) implies that the number of sites on a cluster of size r_s is also given by $s(r_s) \propto w^x r_s^D$. Since there are practically no sites at distances larger than r_s, $\rho_s(r)$ will decay exponentially to zero for $r > r_s$. We thus approximate $\rho_s(r) = 0$ for $r > r_s$, and therefore the sum in (11) contains only clusters with $s > w^x r^D$. On the other hand, n_s decays exponentially to zero when $r_s > \xi$, yielding an upper limit of order $w^x\xi^D$. For $r < r_s < \xi$, we expect all the clusters to look the same, i.e. $\rho_s(r) \simeq \rho_\infty(r)$. With these simplifying assumptions (which may be withdrawn in a detailed calculation), we thus have

$$\rho_\infty(r) \simeq (G(r) + P_\infty^2) \Bigg/ \left(\sum_{s(r)}^{s(\xi)} sn_s + P_\infty\right). \tag{12}$$

For $d > 6$ we may use $sn_s \propto w^{-1/2} s^{-3/2}$, so that $\Sigma\, sn_s \propto w^{-(1+x)/2}(r^{-D/2} - \xi^{-D/2})$. For $r \ll \xi$ we may neglect $\xi^{-D/2}$ and P_∞, and find that $\rho_\infty(r) \propto w^{(1+x)/2} r^{2-d+D/2}$. Thus, $M(L) = \int^L \mathrm{d}^d r\, \rho_\infty(r) \propto w^{(1+x)/2} L^{2+D/2}$. Comparison with w^xL^D therefore identifies $D = 4$ and $x = 1$, and confirms (9). Moreover, (12) allows us to estimate some corrections to this behaviour. The agreement between this result and our other derivations also confirms 'strong self-similarity' for $d > 6$.

Thus far, we emphasised the behaviour of $M(L)$ for $L \ll \xi$. For $\mathrm{r} \gg \xi$, both $G(r)$ and $\Sigma\, sn_s$ in (12) decay exponentially to zero, and (12) approaches the homogeneous limit, $\rho_\infty \to P_\infty$. The details of the crossover from $\rho_\infty \propto wr^{4-d}$ to $\rho_\infty = P_\infty = 1/(w\xi^2)$ are left for future analyses. We note, however, that one can define a series of crossover lengths,

$$L_k = (w\xi^k)^{2/(d-6+2k)} = (L_w^{d-6}\xi^{2k})^{1/(d-6+2k)}. \tag{13}$$

The two terms in the numerator of (12) become comparable at L_2, the two limiting behaviours of $M(L)$ become comparable at L_1 and those of $g(L)$ become comparable at L_3. There probably exists a range of length scales, below ξ, through which various physical quantities cross over from their self-similar to their homogeneous behaviour.

Clearly, all the physical properties scale according to our self-similar predictions (e.g. $M \propto wL^4$, $g \propto L^{-2}$) for $L < L_1 \propto \xi^{2/(d-4)}$, and according to the homogeneous ones for $L > \xi$. It is not yet clear to us whether the range $L_1 < L < \xi$ represents a third scaling regime, or whether there is a separate cross-over for each property. One would also like to obtain a geometrical interpretation of the lengths L_k.

Our last approach uses the mapping onto the limit $q \to 1$ of the q-state Potts model. In this formulation the Hamiltonian has the form (Priest and Lubensky 1976a, b)

$$\mathscr{H} = -\frac{1}{4}\int (r_0 + k^2) \sum_{i=1}^{q} Q_{ii}(\boldsymbol{k})Q_{ii}(-\boldsymbol{k}) + w \int\int \sum_i Q_{ii}(\boldsymbol{k})Q_{ii}(\boldsymbol{k}')Q_{ii}(-\boldsymbol{k}-\boldsymbol{k}'), \tag{14}$$

with $r_0 \propto (p_c - p)$ and $q \to 1$ for percolation. The upper critical dimensionality of this problem is $d_u = 6$. For $d > 6$, w is a 'dangerous irrelevant variable' (Fisher 1973). The critical properties of the model are described by the Gaussian fixed point, near which one has the simple recursion relations (Priest and Lubensky 1976a, b)

$$r(l) = e^{2l} r, \qquad w(l) = e^{(3-d/2)l} w. \tag{15}$$

In the ordered phase ($p > p_c$) one also has the (non-zero) order parameter $Q(l) = e^{(d-2)l/2} Q$. We now iterate until $r(l) \simeq -1$, i.e. $\xi(l) = e^{-l}\xi = 1$, and use mean field theory to find that $Q(l) \propto -r(l)/w(l)$. The non-trivial dependence of $w(l)$ on l now leads to a cancellation of the d-dependence, and one has $P_\infty \propto Q \propto e^{-2l}/w \propto \xi^{-2}/w$, i.e. the mean field result (Pytte 1979). The arguments presented above give some more insight into the role played by w. For $L \ll \xi$, the renormalisation group transformation will transform L into L/e^l, and the scaling form (5) immediately follows (the second variable, ξ/L_w, represents $w(l)^{2/(6-d)} = (w^2\xi^{6-d})^{1/(6-d)}$).

All the above results should be modified at $d = 6$, when logarithmic corrections must be added to most of the power laws. In particular, the correlation function now has the scaling form (Wilson and Kogut 1974)

$$G(\boldsymbol{k}, r, w) = \exp\left(2l - \int_0^l \eta(l')\,dl'\right) G(e^l \boldsymbol{k}, r(l), w(l)), \tag{16}$$

and the iteration is stopped when $r(l) + e^{2l}k^2 = 1$ (Nelson 1976). Since $\eta(l) \propto w(l)^2$, the prefactor yields a logarithmic correction at $d = 6$ (η is of order ε at $d = 6 - \varepsilon$ dimensions). Similarly, logarithmic corrections are induced into sn_s, yielding logarithmic factors into the denominator of (12). A detailed analysis (Kapitulnik *et al* 1983) finally yields

$$M(L, \xi) = \begin{cases} wL^4(\ln L)^{-10/21}, & L \ll \xi, \\ w^{-1}L^6\xi^{-2}(\ln \xi)^{11/21}, & L \gg \xi. \end{cases} \tag{17}$$

Again, these two limiting expressions become comparable at

$$L_0 \simeq w\xi(\ln \xi)^{-1/2} < \xi.$$

Finally, we comment that similar arguments may be applied to many other phase transitions. For example, Ising model clusters have the fractal dimensionality $D_I = d - \beta_I/\nu_I$ for $d < 4$ (Bruce and Wallace 1981, 1983), but $D_I = 3$ for $d > 4$.

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